

Existence Theory and Regularization of Irregular Linear Equations

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Joint work with **Prof. Nikolay Sidorov (ISU)**, Ildar Muftahov (IRNITU),
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Talk Outline

- 1 Perturbation Theory for the First Kind Equations Regularization
- 2 Existence Theory of VIE with Discontinuous Kernels
- 3 Regularization of Cauchy Problem with Non-invertible Operator

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Problem Statement

$$Ax = f, \quad A: X \rightarrow Y \quad (1)$$

$$\tilde{A}x = \tilde{f} \quad (2)$$

$$Ax_\alpha + B(\alpha)x_\alpha = f \quad (3)$$

stabilizing operator $B(\alpha)$ makes solution x_α unique and provides computations stability.

Let us call $\alpha \in S \subset \mathbb{R}^n$ as vector parameter of regularization. Here S is an open set, zero belongs to the boundary of this set (briefly, S -sectorial neighborhood of zero in \mathbb{R}^n), $\lim_{S \ni \alpha \rightarrow 0} B(\alpha) = 0$.

Parameter α we coordinate with input data error level δ

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$$(\tilde{A}x + B(\alpha))\tilde{x}_\alpha = \tilde{f}. \quad (4)$$

Equation (4) we call **regularized equation (RE)** for the problem (2).

$$\|(A + B(\alpha))^{-1}\| \leq c(|\alpha|), \quad (5)$$

$$\|B(\alpha)\| \leq d(|\alpha|), \quad (6)$$

$c(|\alpha|)$ is continuous function, $\alpha \in S \subset \mathbb{R}^n$, $0 \in \bar{S}$,
 $\lim_{|\alpha| \rightarrow 0} c(|\alpha|) = \infty$, $\lim_{|\alpha| \rightarrow 0} d(|\alpha|) = 0$.

let x^* be solution to eq. (1), then

$$x_\alpha - x^* = -(A + B(\alpha))^{-1}B(\alpha)x^*$$

$$S(\alpha, x^*) := \|x_\alpha - x^*\|$$

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Let x^* be some solution to the equation (1), $x(\alpha)$ satisfy the equation (3). Then, in order to $x_\alpha \rightarrow x^*$ for $S \ni \alpha \rightarrow 0$ it is necessary and sufficient to have

$$S(\alpha, x^*) \rightarrow 0 \text{ for } S \ni \alpha \rightarrow 0. \quad (7)$$

Definition

Condition (7) is *stabilization condition*.

Operator $B(\alpha)$, we call *stabilization operator* if it satisfy condition (7).

Solution x^* we call *B-normal solution* of equation (1).

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Lemma (1)

Let x_α and \tilde{x}_α be solutions of the equations (3) and (4) correspondingly.
If parameter $\alpha = \alpha(\delta) \in S$ is selected such as $\delta \rightarrow 0$

$$|\alpha(\delta)| \rightarrow 0 \text{ and } \delta c(|\alpha(\delta)|) \rightarrow 0, \quad (8)$$

then $\lim_{\delta \rightarrow 0} \|x_\alpha - \tilde{x}_\alpha\| = 0$.

Definition

Condition (8) we call the *coordination condition* of vector parameter α with error level δ .

Theorem (1: Main)

Let parameter α be coordinated with noise level δ . Then RE (4) has a unique solution \tilde{x}_α . The following estimate is fulfilled

$$\|\tilde{x}_\alpha - x^*\| \leq S(\alpha, x^*) + \frac{\delta c(|\alpha|)}{1 - q} \left(1 + \|x^*\| + S(\alpha, x^*) \right) \quad (9)$$

where x^* is solution of the exact equation (1). If also x^* is B -normal solution of the equation (1) then $\{\tilde{x}_\alpha\}$ converges to x^* at a rate determined by bound (9) as $\delta \rightarrow 0$.

Remark

For practical applications of this Theorem one needs recommendations on the choice of SO $B(\alpha)$ and conditions for the existence of B -normal solution. It's also useful to know the necessary and sufficient conditions of the existence of B -normal solutions x^* to the exact equation (1).

Theorem (2: Sufficient cond. for existence of B -normal solution)

Let $\|(A + B(\alpha))^{-1}\| \leq c(|\alpha|)$, $\|B(\alpha)\| \leq d(|\alpha|)$ for $\alpha \in S \subset \mathbb{R}^n$,
 $c(|\alpha|)$, $d(|\alpha|)$ are continuous functions, $\lim_{|\alpha| \rightarrow 0} c(|\alpha|) = \infty$,

$\lim_{|\alpha| \rightarrow 0} d(|\alpha|) = 0$. Let $\lim_{|\alpha| \rightarrow 0} c(|\alpha|)d(|\alpha|) < \infty$, $\mathcal{N}(A) = \{0\}$, $\overline{R(A)} = Y$.

Then unique solution x^* of equation (1) is B -normal solution and operator $B(\alpha)$ is its SO.

Theorem (3: Cond. Th. 2 relaxation)

Let $\|(A + \alpha B)^{-1}\| \leq c(\alpha)$, where $\alpha \in \mathbb{R}^1$, $c(\alpha) : (0, \alpha_0] \rightarrow \mathbb{R}^+$ is continuous function. Suppose that there is a positive integer $n \geq 1$ such as $\lim_{\alpha \rightarrow 0} c(\alpha)\alpha^i = \infty$, $i = \overline{0, n-1}$, $\lim_{\alpha \rightarrow 0} c(\alpha)\alpha^n < \infty$. Let x_0 satisfies the equation (1) and in case of $n \geq 2$ there exist x_1, \dots, x_{n-1} which satisfy the sequence of equations $Ax_i = Bx_{i-1}$, $i = \overline{1, \dots, n-1}$. Then x_0 is B -normal solution to equation (1) iff $Bx_{n-1} \in \overline{R(A)}$.

2nd kind equation: A is fredholm operator

If A is fredholm operator, $\{\phi_i\}_1^n$ is basis in $\mathcal{N}(A)$, $\{\psi_i\}_1^n$ is basis in $\mathcal{N}(A^*)$, then we don't regularization parameter.

$B \equiv \sum_{i=1}^n \langle \cdot, \gamma_i \rangle z_i$, where $\{\gamma_i\}$, $\{z_i\}$ are selected such as

$$\det[\langle \phi_i, \gamma_k \rangle]_{i,k=1}^n \neq 0, \langle z_i, \psi_i \rangle = \begin{cases} 1 & \text{if } i = k \\ 0 & \text{if } i \neq k \end{cases}$$

herewith the equation

$$Ax = f - \sum_{i=1}^n \langle f, \psi_i \rangle z_i \quad (10)$$

is resolvable for arbitrary f

Then¹ regularized eq. $\tilde{A}x + \sum_{i=1}^n \langle x, \gamma_i \rangle z_i = \tilde{f} - \sum_{i=1}^n \langle \tilde{f}, \psi_i \rangle z_i$ has unique solution \tilde{x} such as $\|\tilde{x} - x^*\| \rightarrow 0$ for $\delta \rightarrow 0$, where x^* is unique solution of exact solution of (10) such as $\langle x^*, \gamma_i \rangle = 0, i = \overline{1, n}$.

Such method² is employed to proof convergence of iterative methods for nonlinear equations with parameters appearing in the bifurcation theory.

¹V.A. Trenogin. Functional Analysis. Fizmatlit. 2002

²N. Sidorov, B. Loginov et al. Lyapunov-Schmidt methods in nonlinear analysis and applications. Springer Publ. 2013.

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Existence Theorems for Volterra Integral Equations of the First Kind with Discontinuous Kernels

Problem Statement

VIEq with piecewise continuous kernel

$$\int_0^t K(t,s)x(s)ds = f(t), \quad 0 \leq s \leq t \leq T, \quad f(0) = 0, \quad (11)$$

$$K(t,s) = \begin{cases} K_1(t,s), & t,s \in m_1 \\ \dots & \dots \\ K_n(t,s), & t,s \in m_n \end{cases} \quad \begin{matrix} m_i = \{t,s \mid \alpha_{i-1}(t) < s < \alpha_i(t)\}, \\ \alpha_0(t) = 0, \alpha_n(t) = t, i = \overline{1,n} \end{matrix}$$

$\alpha_i(t), f(t) \in C^1_{[0,T]}$, $K_i(t,s)$ have continuous derivatives w.r.t. t for $t,s \in \overline{m_i}$,
 $K_n(t,t) \neq 0, \alpha_i(0) = 0, 0 < \alpha_1(t) < \alpha_2(t) < \dots < \alpha_{n-1}(t) < t,$
 $\alpha_1(t), \dots, \alpha_{n-1}(t)$ increase at least in the small neighborhood $0 \leq t \leq \tau,$
 $0 < \alpha'_1(0) \leq \dots \leq \alpha'_{n-1}(0) < 1$

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Objectives & Methods

Objective

Our objective is to construct the solution $x(t) \in \mathcal{C}_{(0,T]}$

Applications

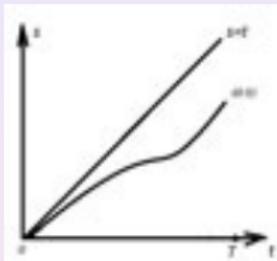
Mathematical models of evolving dynamical systems: vintage capital models, optimal replacement of equipment under technological change, rational harvesting of biological populations.

Methods

We employ the theory of functional equations^a, power-logarithmic asymptotic expansions, the method of steps from delay ODE theory and successive approximations. For numerical solution the quadrature methods are used.

^aGelfond A.O. *The Calculus of Finite Differences*. 5th Edt, Moscow: URSS Publ., 2012

Previous Results



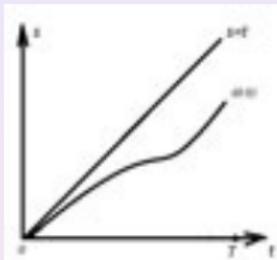
$$\int_{\alpha(t)}^t K(t,s)x(s) ds = f(t), \quad 0 \leq s \leq t \leq T$$

Denisov A. and Lorenzi A. (1995) *On a special Volterra integral equation of the first kind*. Boll. Un. Mat. Ital. B (7) 9, 443-457

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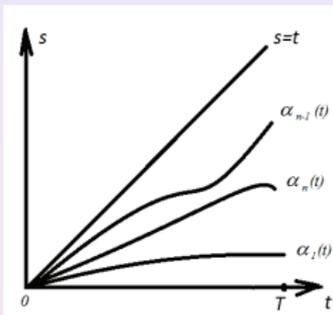
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New Problem Statement

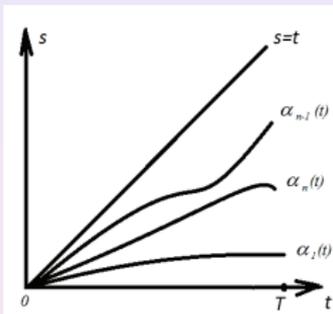


$$\int_0^{\alpha_1(t)} K_1(t, s)x(s) ds + \int_{\alpha_1(t)}^{\alpha_2(t)} K_2(t, s)x(s) ds \cdots + \int_{\alpha_{n-1}(t)}^t K_n(t, s)x(s) ds = f(t)$$

Sidorov D. (2011) *VIE with discontinuous kernels in the theory of evolving systems control*. Studia Informatica Universalis. Vol. 9 (3). P. 135–146

Sidorov D.N. (2013) *Solution to VIE with piecewise smooth kernels*. Differential Equations. 2013. Vol. 49 (2). P. 209–215

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Example 1

$$K(t, s) = \begin{cases} 1, & 0 \leq s < t/2 \\ -1, & t/2 \leq s \leq t \end{cases}$$

$$\int_0^{t/2} x(s) ds - \int_{t/2}^t x(s) ds = t$$

$$x\left(\frac{t}{2}\right) - x(t) = 1, \quad x(t) = c - \frac{\ln t}{\ln 2}$$

Existence and Uniqueness of the Local Solution

$$\begin{aligned} K_n(t, t)x(t) + \sum_{i=1}^{n-1} \alpha_i'(t) \left\{ K_i(t, \alpha_i(t)) - K_{i+1}(t, \alpha_i(t)) \right\} x(\alpha_i(t)) + \\ + \sum_{i=1}^n \int_{\alpha_{i-1}(t)}^{\alpha_i(t)} \frac{\partial K_i(t, s)}{\partial t} x(s) ds - f'(t) = 0 \end{aligned} \quad (12)$$

Reduction to the VIE of the 2nd kind

$$x(t) + \mathbf{A}x + \mathbf{Q}x = \hat{f}(t) \quad (13)$$

A is the functional perturbation operator:

$$\mathbf{A}x \stackrel{\text{def}}{=} K_n^{-1}(t, t) \sum_{i=1}^{n-1} \alpha'_i(t) \left\{ K_i(t, \alpha_i(t)) - K_{i+1}(t, \alpha_i(t)) \right\} x(\alpha_i(t))$$

B is the Volterra operator:

$$\mathbf{Q}x := \int_0^t Q(t, s)x(s)ds \stackrel{\text{def}}{=} \sum_{i=1}^n \int_{\alpha_{i-1}(t)}^{\alpha_i(t)} K_n^{-1}(t, s) \frac{\partial K_i(t, s)}{\partial t} x(s)ds$$

$$\hat{f}(t) \stackrel{\text{def}}{=} K_n^{-1}(t, t)f'(t), \quad K_n(t, t) \neq 0$$

Objective: To obtain the sufficient conditions for existence of local solution, i.e. $\|\mathbf{A} + \mathbf{Q}\| \leq q, q < 1, t \in (0, \tau]$

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$$D(t) \stackrel{\text{def}}{=} \sum_{i=1}^{n-1} \left| \alpha'_i(t) K_n^{-1}(t, t) \right| \cdot \left| K_i(t, \alpha_i(t)) - K_{i+1}(t, \alpha_i(t)) \right|$$

The idea is to use an equivalent norm $\|x\|_l := \sup_{0 < t < \tau} e^{-lt} |x(t)|$, so that the Volterra integral operator \mathbf{Q} becomes contractive.

$$\|\mathbf{Q}x\|_l \leq q(l) \|x\|_l, \quad \lim_{l \rightarrow \infty} q(l) = 0,$$

If $D(0) < 1$, then $\forall q_1 < 1 \exists \tau > 0 D(t) \leq q_1, t \in [0, \tau]$.

Then $\|\mathbf{A}x\|_l \leq q_1 \|x\|_l$

$$\Rightarrow \|\mathbf{A}x\|_l + \|\mathbf{Q}x\|_l \leq (q_1 + q(l)) \|x\|_l \leq q \|x\|_l, \quad 0 < q < q_1 < 1$$

$$\forall l \geq l(q), \quad t \in [0, \tau]$$

Existence & Uniqueness of Local Solution

Theorem 1:

Sufficient Conditions of Existence & Uniqueness of Local Solution

Let for $t \in [0, T]$ the following conditions be fulfilled: continuous $K_i(t, s)$, $i = \overline{1, n}$, $\alpha_i(t)$ and $f(t)$ have continuous derivatives wrt t , $K_n(t, t) \neq 0$, $0 = \alpha_0(t) < \alpha_1(t) < \dots < \alpha_{n-1}(t) < \alpha_n(t) = t$ for $t \in (0, T]$, $\alpha_i(0) = 0$, $f(0) = 0$, $D(0) < 1$, then $\exists \tau > 0$ such as eq. (11) has a unique local solution in $\mathcal{C}_{[0, \tau]}$.

back to example 1:

$$\int_0^{t/2} x(s) ds - \int_{t/2}^t x(s) ds = t, D(t) \equiv 1 \Rightarrow \text{no unique solution!}$$

Existence & Uniqueness of Global Solution

Th. 2: Sufficient Conditions of Existence & Uniqueness of the Global Solution*

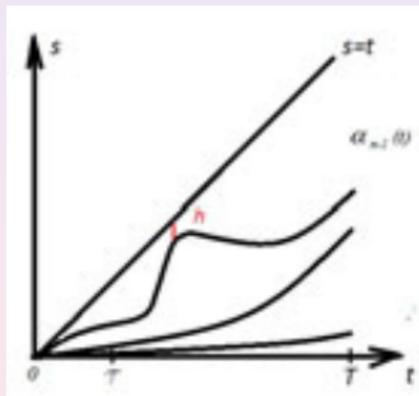
Let the conditions of the Theorem 1 are fulfilled, and moreover let

$$\min_{\tau \leq t \leq T} (t - \alpha_{n-1}(t)) = h > 0.$$

Then eq. (11) has **unique solution** in $C_{[0, T]}$.

*D.N.Sidorov, E.V.Markova. *On One Integral Volterra Model of Developing Dynamical Systems*. Automation and Remote Control. March 2014, Volume 75, Issue 3, pp 413-421.

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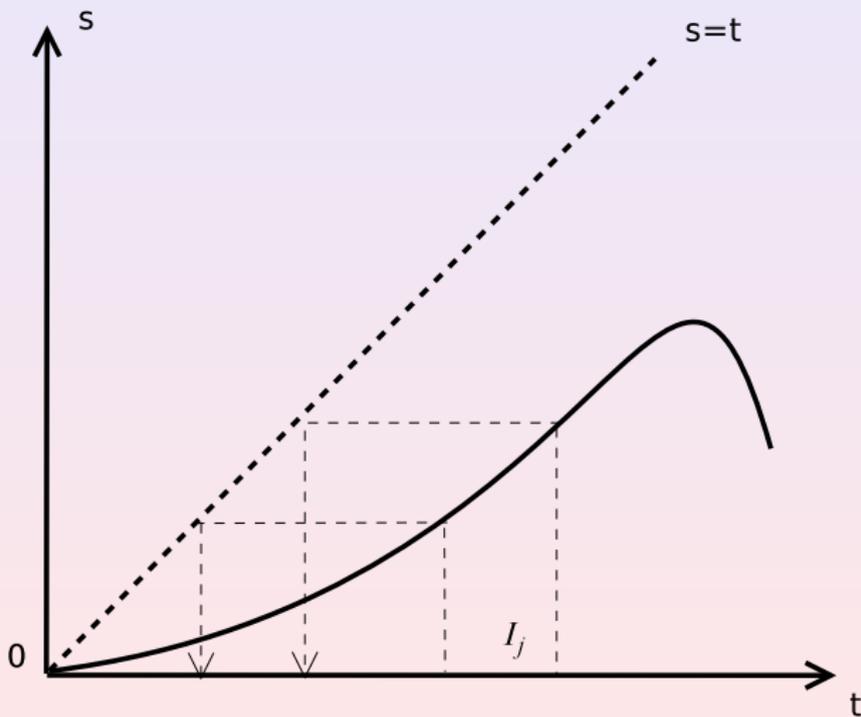


Lemma (1)

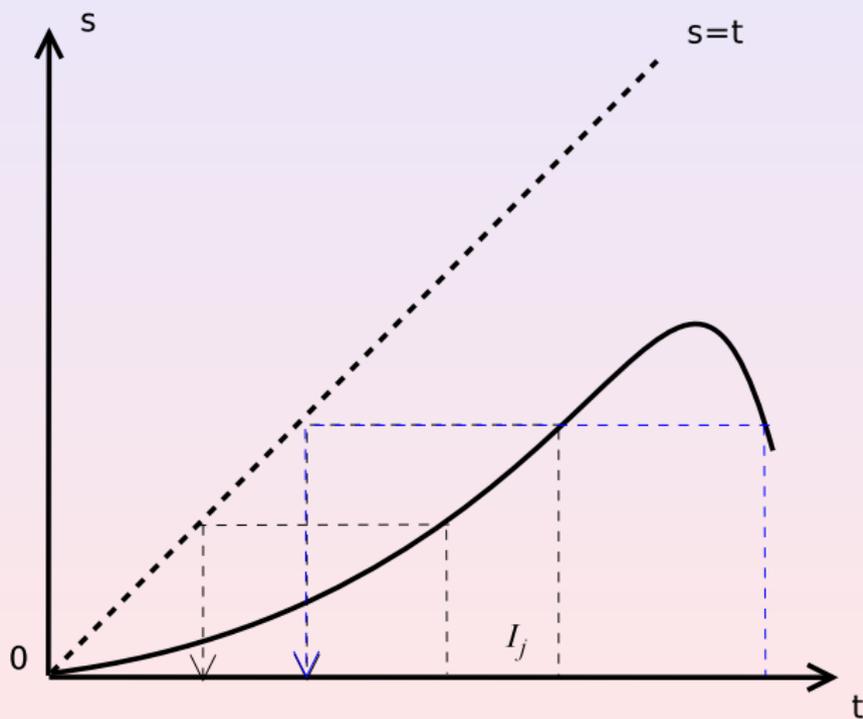
Let $\alpha : [0, T] \rightarrow \mathbb{R}^+$, $\alpha(t) \in C^1_{[0, T]}$, $\alpha(0) = 0$, $0 \leq \alpha^{(1)}(0) < 1$. Let $h \in (0, T)$, $I_0 := [0, h], \dots, I_k := [(1 + (k - 1)\varepsilon)h, (1 + k\varepsilon)h]$, $\varepsilon > 0$, $[0, T] = \bigcup_{k=0}^m I_k$. If $0 < \alpha(t) < t$, $\alpha^{(1)}(t) \leq \frac{1}{1+\varepsilon}$ for $0 \leq t \leq T$ then

$$\alpha : I_j \rightarrow \bigcup_{k=0}^{j-1} I_k, j = \overline{1, m}. \quad (14)$$

Lemma



Lemma



Supplementary Condition of Local Smoothness

So, in case of $D(0) < 1$ the solution exists and unique: local solution is constructed by successive approximations and continued by method of steps with successive approximations on each step on the whole interval.

Therefore theoretical interest is to study the case $D(0) \geq 1$

A. Exists polynomial $\mathcal{P}_i(t, s) = \sum_{\nu+\mu=0}^M K_{i\nu\mu} t^\nu s^\mu, i = \overline{1, n},$

$f^M(t) = \sum_{\nu=1}^M f_\nu t^\nu, \alpha_i^M(t) = \sum_{\nu=1}^M \alpha_{i\nu} t^\nu, i = \overline{1, n-1},$ where

$0 < \alpha_{11} < \alpha_{12} < \dots < \alpha_{n-1, n} < 1$ such as for $t \rightarrow +0, s \rightarrow +0$ the following estimates hold:

$$\begin{aligned}
 |K_i(t, s) - \mathcal{P}_i(t, s)| &= \mathcal{O}((t+s)^{M+1}), i = \overline{1, n}, \\
 |f(t) - f^M(t)| &= \mathcal{O}(t^{M+1}), \\
 |\alpha_i(t) - \alpha_i^M(t)| &= \mathcal{O}(t^{M+1}), i = \overline{1, n-1}.
 \end{aligned}$$

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$0 < \alpha_{11} < \alpha_{12} < \dots < \alpha_{n-1, n} < 1$ such as for $t \rightarrow +0, s \rightarrow +0$ the following estimates hold:

$$|K_i(t, s) - \mathcal{P}_i(t, s)| = \mathcal{O}((t + s)^{M+1}), i = \overline{1, n},$$

$$|f(t) - f^M(t)| = \mathcal{O}(t^{M+1}),$$

$$|\alpha_i(t) - \alpha_i^M(t)| = \mathcal{O}(t^{M+1}), i = \overline{1, n-1}.$$

Supplementary Condition of Local Smoothness

Let $0 \leq \alpha'_i(0) < 1$, $\alpha_i(0) = 0$, $i = \overline{1, n-1}$. Then $\forall \varepsilon \in (0, 1)$
 $\exists \tau \in (0, T] : \max_{i=\overline{1, n-1}, t \in [0, \tau]} |\alpha'_i(t)| \leq \varepsilon, \quad \sup_{t \in (0, \tau]} \frac{\alpha_{n-1}(t)}{t} \leq \varepsilon.$

B. For fixed $q \in (0, 1)$, $\tau \in (0, T]$, $0 < \varepsilon < 1$

$$\max_{t \in [0, \tau]} \varepsilon^M D(t) \leq q < 1$$

Estimate is true for big enough M

Approximation and Regularization of the Solution

C. Select N^* such as $\left(\int_0^t K(t, s) \widehat{x}(s) ds - f(t) \right)' = \mathcal{O}(t^{N^*})$

Theorem* 3

Let $\widehat{x}(t)$ be the known function such as the discrepancy \mathbf{C} is true for $N^* \geq M$. Then eq. (11) has the solution $x(t) = \widehat{x}(t) + t^{N^*} u(t)$, where $u(t) \in \mathcal{C}_{[0, T]}$ is unique and can be constructed by means of the successive approximations method.

*D.Sidorov. Solution to VIE with piecewise smooth kernels. Differential Equations. 2013. Vol. 49 (2). P. 209-215.

Regularization

For $u(t)$ we have the equation

$$\int_0^t K(t, s) s^N u(s) ds = g(t) \quad (*)$$

$$g(t) := - \int_0^t K(t, s) \widehat{x}(s) ds + f(t),$$

$$g(t) \in \mathcal{C}_{[0, T]}^{(1)}, g(0) = 0, |g'(t)| = o(t^N), \text{ as } t \rightarrow +0$$

Definition

The equation (*) has unique solution and we call it as *regularization* of the equation (11). Function $\widehat{x}(t)$ is approximation of solution to eq. (1)

Question Remains:

How to Construct an Approximation $\hat{x}(t)$ to meet the Condition **C**?

Idea:

To construct $\hat{x}(t)$ as power-logarithmic asymptotic expansion.
We would need an additional smoothness of all the given functions according to the condition **A** since we need the Taylor coefficients.

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Construction of the Power-Logarithmic Asymptotic of Solution

We search for the approximation of solution as $\widehat{x}(t) = \sum_{j=0}^N x_j t^j$.

To find x_j using method of undetermined coefficients we must solve the recurrence sequence of linear algebraic equations

$$B(j)x_j = \mathcal{M}_j(x_0, \dots, x_{j-1}), \quad \mathcal{M}_0 = f'(0)$$

Characteristic Equation

$$B(j) \equiv K_n(0,0) + \sum_{i=1}^{n-1} (\alpha'_i(0))^{1+j} (K_i(0,0) - K_{i+1}(0,0)) = 0$$

$B(j) \neq 0, j \in \mathbb{N} \cup \{0\}$ — **regular case**: x_j are constant coefficients

$B(j) = 0, j \in \mathbb{N} \cup \{0\}$ — **singular case**: x_j are polynomials of $\ln t$.

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Singular Case

seek for the solution as $\widehat{x}(t) = \sum_{j=0}^N x_j(\ln t) t^j$

coeff. x_{j^*} is the solution of difference equation

$$\begin{aligned} K_n(0,0)x_{j^*}(z) + \sum_{i=1}^{n-1} (\alpha'_i(0))^{1+j^*} (K_i(0,0) - K_{i+1}(0,0)) \cdot x_{j^*}(z + \ln \alpha'_i(0)) = \\ = \mathcal{M}_j(x_0(z), \dots, x_{j^*-1}(z)), \quad z := \ln t \end{aligned} \quad (15)$$

Example 1

Back to example 1: $\int_0^{t/2} x(s) ds - \int_{t/2}^t x(s) ds = t$

Characteristic eq. $B(j) \equiv -1 + (1/2)^j = 0$, root: $j^* = 0 \Rightarrow$

$$x\left(\frac{t}{2}\right) - x(t) = 1,$$

$$x(t) = C_1 + C_2 \ln t$$

$$C_1 + C_2 \left(\ln \frac{t}{2}\right) - C_1 - C_2 \ln t = 1 \Rightarrow C_2 = -\frac{1}{\ln 2}$$

Solution:

$$x(t) = -\frac{\ln t}{\ln 2} + \text{const.}$$

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Condition $D(0) < 1$ relaxation

Th 4 (Relaxed suff. cond. of ex. and uniq.)

Let cond. **(B)** and **(C)** be satisfied,

let $B(j) := K_n(0, 0) + \sum_{i=1}^{n-1} (\alpha'_i(0))^{1+j} (K_i(0, 0) - K_{i+1}(0, 0)) \neq 0$ for
 $j \in \mathbb{N} \cup \{0\}$. Then eq. (11) has unique solution $x(t) = x^M(t) + t^{N^*} u(t)$
in $C_{[0, T]}$, $M \geq N$. Then as $t \rightarrow +0$ polynomial $\hat{x}(t) \equiv x^M(t) = \sum_{i=0}^M x_i t^i$ is
 M th order asymptotic of such solution.

Generalization on the case of piecewise continuous source

$$\mathcal{K}x := \sum_{i=1}^n \int_{\alpha_{i-1}(t)}^{\alpha_i(t)} K_i(t, s)x(s) ds = f(t), \quad t \in \mathbb{R}^1, \quad t < T < \infty,$$

$$f(t) = \begin{cases} f_1(t), & -\infty < t < T_1, \\ f_2(t), & T_1 < t < T_2, \\ \dots & \dots \\ f_m(t), & T_{m-1} < t < T < \infty, \end{cases} \quad 0 < T_1 < T_2 < \dots < T_{m-1}.$$

$f_i(t)$ are defined, continuous and differentiable for $T_{i-1} < t < T_i$,
 $T_0 = 0, T_m = T$.

$$x(t) = \sum_{k=1}^{m-1} c_k \delta(t - T_k) + u(t)$$

reducing such eq. to scalar VIE with cont. source function, where $\delta(t - T_k)$ is Dirac function, $u(t)$ is regular function.

$$\mathbf{c} = \mathfrak{A}^{-1}\mathbf{b},$$

$$\mathfrak{A} = \begin{pmatrix} K_1(T_1, T_1) & 0 & 0 & \dots & 0 \\ K_1(T_2, T_1) & K_1(T_2, T_2) & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ K_1(T_{m-1}, T_1) & K_1(T_{m-1}, T_2) & \dots & \dots & K_1(T_{m-1}, T_{m-1}) \end{pmatrix},$$

$$\mathbf{b} = \begin{pmatrix} f_2(T_1) - f_1(T_1) \\ f_3(T_2) - f_2(T_2) \\ \dots \\ f_m(T_{m-1}) - f_{m-1}(T_{m-1}) \end{pmatrix}$$

Example. $\int_{t/2}^t x(s) ds = \begin{cases} 0, & t < 1, \\ t-1, & t > 1 \end{cases}$ has solution

$$x^*(t) = \begin{cases} 0, & -\infty < t < 1, \\ 1, & 1 < t < 2, \\ 1 + 1/2, & 2 < t < 2^2, \\ \dots \\ 2 - \left(\frac{1}{2}\right)^n, & 2^{n-1} < t < 2^n, \\ \dots \end{cases} \quad \sup_{-\infty \leq t \leq \infty} x^*(t) = 2.$$

$$\mathbf{c} = \mathfrak{A}^{-1}\mathbf{b},$$

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Solution to the nonlinear VIE with piecewise continuous kernels

$$\int_0^t K(t, s, x(s)) ds = f(t), \quad 0 \leq s \leq t \leq T, \quad f(0) = 0 \quad (16)$$

$$K(t, s, x(s)) := \begin{cases} K_1(t, s)G_1(s, x(s)), & t, s \in m_1, \\ \dots & \dots \\ K_n(t, s)G_n(s, x(s)), & t, s \in m_n, \end{cases}$$

$m_i = \{t, s \mid \alpha_{i-1}(t) < s < \alpha_i(t)\}$, $\alpha_0(t) = 0$, $\alpha_n(t) = t$, $i = \overline{1, n}$, $K_i(t, s)$, $f(t)$, $\alpha_i(t)$ have continuous derivatives wrt t as $t, s \in \overline{m_i}$, $K_n(t, t) \neq 0$, $\alpha_i(0) = 0$, $0 < \alpha_1(t) < \alpha_2(t) < \dots < \alpha_{n-1}(t) < t$, $\alpha_1(t), \dots, \alpha_{n-1}(t)$ increases at least in the small neighborhood $0 \leq t \leq \tau$.

$$\begin{aligned} \text{(D)} \quad & \|G_i(s, x_1(s)) - G_i(s, x_2(s)) - (x_1(s) - x_2(s))\| \leq q_i \|x_1 - x_2\|, \forall x_1, x_2, \\ \text{(E)} \quad & q_n + \sum_{i=1}^{n-1} \alpha'_i(0) \|K_n(0, 0)^{-1}(K_i(0, 0) - K_{i+1}(0, 0))\| (1 + q_i) < 1. \end{aligned}$$

Th 2' (Suff. cond. of ex. and uniq. of NVIE)

Let cond. (D) and (E) are fulfilled as $0 \leq s \leq t \leq T$. Then $\exists \tau > 0$ such as eq. (16) has unique local solution in $\mathcal{C}_{[0, \tau]}$. If

$\min_{\tau \leq t \leq T} (t - \alpha_{n-1}(t)) = h > 0$, then such local solution one can continue on the whole $[\tau, T]$ using step method and successive approximations.

Generalizations:

Systems of linear VIEs with piecewise continuous kernels:

Denis Sidorov. *Solvability of system of the VIEs of the 1st kind with piecewise continuous kernels*. Russian Mathematics. – 2013. – Vol. 57(1). – P.54-63.

Abstract integral-operator Volterra equations in Banach spaces:

Alfredo Lorenzi. *Integral-differential operator equations of the first kind of degenerate type in Banach spaces and applications to integral-differential PDEs*. Eurasian Journal of Mathematical and Computer Applications. Vol. 4 2013

Nikolay Sidorov and Denis Sidorov. *On the solvability of a class of Volterra operator equations of the first kind with piecewise continuous kernels*. Mathematical Notes. Vol. 96 (5-6) 2014, pp 811-82. **arXiv:1208.3604**

Numerical Method

$\Omega_x^N := \{t_i | t_i = i/N, i = 0, \dots, N\}$, also nonuniform meshes can be used
 $0 = t_0 < t_1 < t_2 < \dots < t_N = T$, $h = \max_{i=1, N} (t_i - t_{i-1}) = \mathcal{O}(N^{-1})$. we

seek for the solution as piecewise constant function

$$x_N(t) = \sum_{i=1}^N x_i \delta_i(t), \quad t \in (0, T],$$

$$\delta_i(t) = \begin{cases} 1, & \text{for } t \in \Delta_i = (t_{i-1}, t_i] \\ 0, & \text{for } t \notin \Delta_i \end{cases}$$

with coefficients x_i , $i = \overline{1, N}$.

D. Sidorov, A. Tynda and I. Muftahov. *Numerical Solution of Weakly Regular Volterra Integral Equations of the First Kind*. arXiv:1403.3764v2.

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$$f'(t) = \sum_{i=1}^n \left(\int_{\alpha_{i-1}(t)}^{\alpha_i(t)} \frac{\partial K_i(t, s)}{\partial t} x(s) ds + \alpha'_i(t) K_i(t, \alpha_i(t)) x(\alpha_i(t)) - \alpha'_{i-1}(t) K_i(t, \alpha_{i-1}(t)) x(\alpha_{i-1}(t)) \right).$$

$$x_0 = \frac{f'(0)}{\sum_{i=1}^n K_i(0, 0) [\alpha'_i(0) - \alpha'_{i-1}(0)]}. \quad (17)$$

Based Th. 1

$$\sum_{i=1}^n K_i(0, 0) \cdot [\alpha'_i(0) - \alpha'_{i-1}(0)] \neq 0.$$

Let's make the notation: $f_k := f(t_k)$, $k = 1, \dots, N$. In order to define the coefficient x_1 we write our equation in the point $t = t_1$

$$\sum_{i=1}^n \int_{\alpha_{i-1}(t_1)}^{\alpha_i(t_1)} K_i(t_1, s)x(s)ds = f_1. \quad (18)$$

Since on this step $\alpha_i(t_1) - \alpha_{i-1}(t_1) \leq h$ we have

$$x_1 = \frac{f_1}{\sum_{i=1}^n (\alpha_i(t_1) - \alpha_{i-1}(t_1)) K_i(t_1, \frac{\alpha_i(t_1) + \alpha_{i-1}(t_1)}{2})}. \quad (19)$$

Let v_{ij} be number of interval of the mesh Ω_x^N , on which the value $\alpha_i(t_j)$ is falling, i.e. $\alpha_i(t_j) \in \Delta_{v_{ij}}$.

Let's the coefficients x_0, x_1, \dots, x_{k-1} of the approximate solution are defined. In the point $t = t_k$ our VIE

$$\sum_{i=1}^n \int_{\alpha_{i-1}(t_k)}^{\alpha_i(t_k)} K_i(t_k, s)x(s)ds = f_k,$$

can written as follows $I_1(t_k) + I_2(t_k) + \dots + I_n(t_k) = f_k$, where

$$I_1(t_k) := \sum_{j=1}^{v_{1,k}-1} \int_{t_{j-1}}^{t_j} K_1(t_k, s)x(s) ds + \int_{t_{v_{1,k}-1}}^{\alpha_1(t_k)} K_1(t_k, s)x(s) ds,$$

...

$$I_n(t_k) := \int_{\alpha_{n-1}(t_k)}^{t_{v_{n-1,k}}} K_n(t_k, s)x(s) ds + \sum_{j=v_{n-1,k}+1}^k \int_{t_{j-1}}^{t_j} K_n(t_k, s)x(s) ds.$$

Remark. Number of terms in l_i depends on v_{ij} which is defined on input data: functions $\alpha_i(t)$, $i = \overline{1, n-1}$, and fixed mesh with certain N .

$$\int_{t_{v_p, k-1}}^{\alpha_p(t_k)} K_p(t_k, s)x(s)ds \approx$$

$$\approx (\alpha_p(t_k) - t_{v_p, k-1}) K_p \left(t_k, \frac{\alpha_p(t_k) + t_{v_p, k-1}}{2} \right) x_N \left(\frac{\alpha_p(t_k) + t_{v_p, k-1}}{2} \right).$$

On the intervals where desired function is known we select $x_N(t)$ (i.e. $t \leq t_{k-1}$). On the rest of the intervals unknown x_k appears in the last terms.

$$\varepsilon = \max_{0 \leq i \leq N} |\bar{x}(t_i) - x^h(t_i)|$$

Numerical method enjoy order of $\mathcal{O}(N^{-1})$.

Example

$$\int_0^{t/3} (1+t-s)x(s) ds - \int_{t/3}^t x(s) ds = \frac{t^4}{108} - \frac{25t^3}{81}, \quad t \in [0, 2] \quad \bar{x}(t) = t^2$$

$\varepsilon_N = \|x^N(t_i) - \bar{x}(t_i)\|_{\Omega^N}$, order of convergence $p^N = \log_2 \frac{D^N}{D^{2N}}$, where
 $D^N = \|x^N(t_i) - \bar{x}^{2N}(t_i)\|_{\Omega^N}$.

	32	64	128	256	512	1024	2048
ε_N	0.1303	0.0780	0.0398	0.0197	0.0100	0.0050	0.0025
D^N	0.07462	0.0381	0.0201	0.0097	0.0051	0.0025	0.0012
p^N	0.9677	0.9220	1.0461	0.9221	0.9886	1.0071	0.98639

Example

$$\int_0^{\frac{t}{9}} (1+t-s)x(s) ds - \int_{\frac{t}{9}}^{\frac{2t}{9}} x(s) ds - 2 \int_{\frac{2t}{9}}^{\frac{4t}{9}} x(s) ds + \int_{\frac{4t}{9}}^t x(s) ds = \frac{11t^4}{26244} + \frac{547t^3}{2187},$$

$t \in [0, 2]$, $\bar{x}(t) = t^2$ is exact solution. $\varepsilon_N = \|x^N(t_i) - \bar{x}(t_i)\|_{\Omega^N}$, order of convergence $p^N = \log_2 \frac{D^N}{D^{2N}}$, where $D^N = \|x^N(t_i) - \bar{x}^{2N}(t_i)\|_{\Omega^N}$.

	32	64	128	256	512	1024	2048
ε_N	0.1371	0.0740	0.0453	0.0221	0.0110	0.0054	0.0027
D^N	0.0903	0.0463	0.0260	0.0141	0.0068	0.0034	0.0016
p^N	0.9611	0.8318	0.8827	1.0342	0.9952	0.0274	0.8685

Generalization for systems of VIE

$$\int_0^{t/4} \begin{pmatrix} 2 + ts, & 1 - ts \\ 1 + t + s, & 1 - ts \end{pmatrix} \cdot \begin{pmatrix} x_1(s) \\ x_2(s) \end{pmatrix} ds + \int_{t/4}^{t/2} \begin{pmatrix} 1 + t + s, & t + s \\ \frac{s}{1+t}, & t + s \end{pmatrix} \cdot \begin{pmatrix} x_1(s) \\ x_2(s) \end{pmatrix} ds +$$

$$+ \int_{t/2}^t \begin{pmatrix} 1 + t + s, & -1 \\ 1, & \frac{-1}{1+t} \end{pmatrix} \cdot \begin{pmatrix} x_1(s) \\ x_2(s) \end{pmatrix} ds = \begin{pmatrix} \frac{t^6}{40960} + \frac{1129t^5}{20480} + \frac{2027t^4}{24576} - \frac{55t^3}{192} \\ \frac{1}{t+1} \cdot \left(\frac{-17t^6}{20480} + \frac{4943t^5}{61440} + \frac{2107t^4}{24576} - \frac{55t^3}{192} \right) \end{pmatrix},$$

$$x(t) = \begin{pmatrix} t^3/8 \\ t^2 \end{pmatrix}, \quad \varepsilon_i = \max_{0 \leq j \leq N} |x_i(t_j) - x_i^{(N)}(t_j)|$$

h	1/32	1/64	1/128	1/256	1/512	1/1024	1/2048
ε_1	0,054205	0,028218	0,014392	0,007267	0,003652	0,001830	0,000916
ε_2	0,127717	0,064389	0,032327	0,016196	0,008106	0,004055	0,002028

Non-classic Cauchy Problem with Non-invertible Operator

Classic Cauchy Problem

$$\begin{cases} B\dot{x}(t) = x(t) + f(t) & (P1) \\ x(0) = x_0 & (P2) \end{cases}$$

$B : X \rightarrow X$ is non-invertible linear bounded operator
e.g. Fredholm

Then Cauchy Problem (P1)-(P2) is not solvable $\forall x_0!$

Necessary conditions for solvability

If (P1) is solvable then it's necessary $\langle x(0) + f(0), \psi \rangle = 0$, where $B^*\psi = 0$

Classic Cauchy Problem is Ill-posed Problem for arbitrary $x_0!$

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$$\begin{cases} B\dot{x}(t) = x(t) + f(t) & (P1) \\ x(0) = x_0 & (P2) \end{cases}$$

$B : X \rightarrow X$ is non-invertible linear bounded operator
e.g. Fredholm

Then Cauchy Problem (P1)-(P2) is not solvable $\forall x_0!$

Necessary conditions for solvability

If (P1) is solvable then it's necessary $\langle x(0) + f(0), \psi \rangle = 0$, where $B^*\psi = 0$

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IDEA: Regularization via non-classic initial condition

Nonclassic Cauchy Problem

$$\begin{cases} B\dot{x}(t) = x(t) + f(t) & (P1) \\ Mx(t)|_{t=0} = c_0 & (P3) \end{cases}$$

Find linear space X_1 and linear operator $M : X \rightarrow X_1$ such as Cauchy problem (P1), (P3) will have unique continuous solution for $\forall c_0 \in X_1$. Solution depends continuously on $c_0 \in X_1$.

How to build operator $M \in \mathcal{L}(X \rightarrow X_1)$?

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Nonclassic Cauchy Problem

Theorem (1)

Let $B = A_1 A_2$, where $A_2 : X \rightarrow X_1$, $A_1 : X_1 \rightarrow X$.

Introduce $B_1 := A_2 A_1 : X_1 \rightarrow X_1$ and presume B_1 is invertible. Let $f(t) \in C^1$. Then problem

$$\begin{cases} B\dot{x}(t) = x(t) + f(t) \\ A_2 x(t)|_{t=0} = c \end{cases}$$

enjoy unique solution for **arbitrary** $c \in X_1$. Solution depends continuously on c .

Nonclassic Cauchy Problem

Remark 1.

If it appear $A_2 A_1 \equiv 0$ then homogeneous system $B\dot{x} = x$ has only trivial solution them $\nexists M$ and initial condition is redundant. $B^2 \equiv 0$, B is nilpotent and eq. (1) has unique solution $x(t) = -f(t) - Bf'(t)$.

Nonclassic Cauchy Problem

Let B_1 is non-invertible.

Let us introduce operators

$$A_{2i} : X_{i-1} \rightarrow X_i, \quad A_{2i-1} : X_i \rightarrow X_{i-1}, \quad B^i \equiv A_{2i}A_{2i-1} : X_i \rightarrow X_i, \quad i = \overline{1, p},$$

where X_i are linear spaces, $X_0 \equiv X$. Let B^0, B^1, \dots, B^{p-1} be noninvertible, and B^p is continuously invertible or zero operator from X_p to X_p . Let also $B^0 = A_1A_2 \equiv B$, $A_{2i}A_{2i-1} = A_{2i+1}A_{2i+2}$, $i = \overline{1, p}$.

Then i -th operator of the sequence $\{B^i\}_{i=0}^p$ is constructed by permutation of skeleton decomposition of operators of $i-1$ th operator of this chain.

Definitions.

The sequence $\{B^i\}_{i=0}^p$ we call **skeleton chain** of finite length of the operator B .

If B^p is invertible then we can that chain as **regular chain**.

If B^p is zero operator then we call the chain $\{B^i\}_{i=0}^p$ as **degenerate chain**. In this case operator B^p is nilpotent.

Nonclassic Cauchy Problem

we can introduce functions $x_i(t) = \prod_{j=1}^i A_{2j} x_0(t)$, where $x_0(t)$ is solution of eq. (1), $x_i(t) = A_{2i} x_{i-1}$, $i = \overline{1, p}$. Let $\{B_i\}_{i=0}^p$ be regular skeleton chain. Then function $x_p(t)$ satisfies the regular Cauchy problem

$$\begin{cases} B^p \frac{dx_p}{dt} = x_p + \prod_{j=1}^p A_{2j} f(t), \\ x_p(0) = c_0. \end{cases}$$

Once we have functions $x_p(t)$ determined, the rest of functions x_{p-1}, \dots, x_0 we construct following recursion

$$x_i = - \prod_{j=1}^i A_{2j} f(t) + A_{2i} \frac{dx_{i+1}}{dt}, \quad i = p-1, \dots, 1,$$

$$x_0 = -f(t) + A_1 \frac{dx_1}{dt}.$$

Theorem (2: Main)

If operator B has regular skeleton chain of length $p + 1$, $f(t)$ is $p + 1$ times differentiable then eq. (1) with initial condition

$$\prod_{j=1}^p A_{2j} x(t) \Big|_{t=0} = c_0, c_0 \in X_p$$

enjoys unique classic solution $x_0(t, c_0)$. If operator B has degenerate skeletal chain of length p then B is nilpotent.

Then in case of $f(t) \in \mathcal{C}^{(p+1)}$ eq. (1) enjoys unique classic solution.

Lyapunov-Schmidt Methods in Nonlinear Analysis and Applications



Nikolay Sidorov, Boris Loginov, A.V. Sinityn, M.V. Fataleev

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0 Отзывов

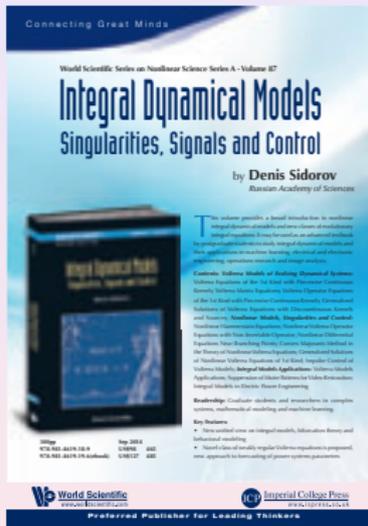
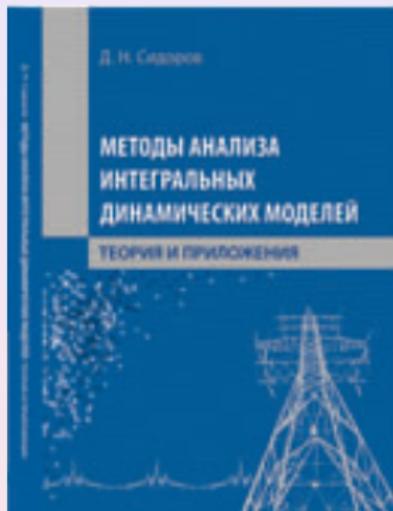
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This book concentrates on the branching solutions of nonlinear operator equations and the theory of degenerate operator-differential equations especially applicable to algorithmic analysis and nonlinear PDE's in mechanics and mathematical physics. The authors expound the recent result on the generalized eigen-value problem, the perturbation method, Schmidt's pseudo-inversion for regularization of linear and nonlinear problems in the branching theory and group methods in bifurcation theory.

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